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Thermodynamics of Black Holes from an Entropy Functional: An Other Approach Using Generalized Elasticity

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Abstract In a series of recent papers Padmanabhan et al. derived Einstein equations for gravity by introducing an entropy functional for space-time viewed as an elastic medium. They showed that the same entropy functional applied to the thermodynamics of horizons yields an entropy that is always proportional to the area of the horizon. Following the same philosophy as theirs we first note that it may be arrived at Einstein equations, with a cosmological constant as an integration constant, using a slightly different route from theirs that also results in the same final expression for the entropy functional. We generalize the fundamental equations of three-dimensional elasticity to four dimensions and propose that the elastic deformation of space-time be constrained by those equations. A general Lagrangian describing the elastic deformation of space-time is deduced. When applied to the special case of a Schwarzschild black hole, viewed as an infinite line defect in space-time, the approach developed here permits to recover the black hole's mass from the elastic deformation it caused to space-time, to reproduce the Hawking temperature, and to yield an entropy that is also in agreement with the Bekenstein-Hawking formula.

Keywords Elastic space-time · Einstein equations · Bekenstein-Hawking entropy formula

1 Introduction

The idea of identifying space-time with an elastic medium goes back to A.D. Sakharov who proposed in 1968 that space-time is the coarse grained limit of some microscopic substructure and that general relativity is just a macroscopic approximation to a microscopic theory of the dynamics of space-time (he calls it the "metric elasticity of space") just as elasticity is to chemical physics [1]. According to this idea it is then interesting to try to extend the classical three-dimensional theory of elasticity to four dimensions and compare the dynamics of the four-dimensional elastic medium it describes to the dynamics of space-time described by Einstein equations. By applying the methods of elasticity theory and adapting some of its

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concepts to space-time, one hopes to bring new light to some of the results of general relativity and propose new interpretations to some of its extensions. This approach, carried out recently by several authors has led, among other things, to the study of dislocation-like solutions to Einstein equations and their relation to cosmic strings [2–4], the study of the motion of photons around space-time dislocations [5, 6], an interpretation of Einstein-Cartan theory in terms of defects in space-time [7], and more remarkably a proposal to explain inflation in terms of a spherical defect in space-time [8–10]. Other authors have also explored the other way around, that is to apply the concepts of general relativity to the physics of solid mediums [11, 12].

An other axis of research combining concepts from solid state physics and the physics of space-time consists of investigating the link between the fundamental concepts of the theory of elastic mediums and the fundamental equations of the dynamics of space-time. Such a link would show that Einstein equations may really be derived from investigating the assumption that space-time is an elastic medium.

Work in this direction has also been conducted by various authors, to cite but a few, see e.g. [13–16]. In [13], Einstein equations were arrived at by investigating the dynamics of a thin and strained four-dimensional elastic plate identified with space-time by introducing extra-dimensions into which the plate bends. In [14], sticking to four dimensions, the author showed by introducing an entropy functional that Einstein equations are sufficient constraints on the dynamics of space-time that allow the latter to be identified with a four-dimensional elastic medium with an extremized entropy. In the same reference the author showed that the same functional implies a proportionality of entropy to the area of any horizon. In [15] and [16], the general case of D dimensions is considered and a novel approach to the derivation of Einstein equations based on null vector fields can be found.

In this paper we shall follow closely the work in [14]. This paper is organized as follows. In Sect. 2, after briefly reviewing how it is arrived at Einstein equations in [14], we show that these equations may also be arrived at by following a slightly different route that produces at once a cosmological constant as an integration constant and ending up with the same final expression for the entropy functional as in [14]. In Sect. 3, we generalize the fundamental equations of equilibrium of three-dimensional elasticity theory to four dimensions and give a general Lagrangian to describe the elastic deformations of space-time. In Sect. 4, we explicitly evaluate the entropy of a Schwarzschild black hole by solving the generalized equations of equilibrium of Sect. 3 then plugging the resulting solution in the entropy functional. In Sect. 5, we apply the results of Sect. 4 to recover the mass of the Schwarzschild black hole from the pure elastic deformation it caused to space-time. We conclude this paper with a discussion on our various results.

2 Einstein Equations from an Entropy Functional Revisited

By identifying space-time with an elastic medium, our starting point to arrive at Einstein equations will be, as in [14], the general form of the entropy of a four-dimensional elastic medium. The fundamental variable in elasticity theory is usually taken to be the deformation vector field [17] $u^i(x) = \tilde{x}^i - x^i$, where $i = 1, \ldots, 3$, and $\tilde{x}^i$ are coordinates labelling points in the deformed medium while x^i are coordinates labelling the same points in the medium before deformation. Then, as it is explained in [14], to lowest order the entropy of the four-dimensional elastic space-time should be a scalar quadratic both in the deformation vector field u^i and its first covariant derivatives $\nabla^i u^j$ $(i, j = 0, \ldots, 3)$. The derivative terms are the space-time contribution to entropy arising even in the absence of external sources, matter and energy. The non-derivative terms arise in the presence of these external sources representing defects in space-time that break translational invariance [15]. The author then took for the entropy functional the following general form answering the above requirements, $\mathcal{S} = (1/8\pi)\int \mathrm{d}^4x\sqrt{-g}[M^{ijkl}\nabla_i u_j \nabla_k u_l + N_{ij}u^i u^j]$, where M^{ijkl} and N_{ij} are two tensors to be determined. Varying $\mathcal{S}$ with respect to the deformation field u^i, the author obtained the following equations of motion $\nabla_i(M^{ijkl}\nabla_k)u_l = N^{jl}u_l$. Then the author states that in the case of elasticity theory one would have used such an equation to determine the deformation field u^i but for the case of space-time one requires *any* deformation u^i to be allowed *provided* the background space-time satisfies Einstein equations. He then shows that the unique choice $M^{ijkl} = g^{il}g^{jk} - g^{ij}g^{kl}$ and $N_{ij} = 8\pi(T_{ij} - g_{ij}T/2)$ really allows for the above equations of motion to be satisfied for any u^i and giving at once Einstein equations. In [15] and [16], this method is generalized to D dimensions and a different expression, with a different motivation, for the tensor M^{ijkl} is given. The entropy functional there is applied to null hypersurfaces and the derivation of Einstein equations is based on null vector fields u^i.

Here we adopt a different route by following different steps. We start with a different expression for the entropy functional and demand that the equations of motion that result from its variation be satisfied for any u^i without seeking the satisfaction of Einstein equations by the background. The general form we take here for the functional representing entropy is

$$\mathcal{S} = \int \mathrm{d}^4x\,\sqrt{-g}[\alpha(\nabla_i u_j)(\nabla^j u^i) + \beta(\nabla_i u_j)(\nabla^i u^j) + \gamma(\nabla_i u^i)^2 + \lambda g_{ij}u^i u^j + T_{ij}u^i u^j], \quad (1)$$

where α, β, γ, λ, and T_{ij} are, respectively, four scalars and a symmetric tensor, to be determined.

Since T_{ij} goes with the factor coming from the contributions of matter and energy, it is natural to identify this tensor with the symmetric energy-momentum tensor associated with matter and energy. On the other hand, the variation of (1) with respect to the deformation vector field u^i gives

$$\begin{aligned}&\nabla_j\alpha\nabla_i u^j + \nabla_j\beta\nabla^j u_i + \nabla_i\gamma\nabla_j u^j + \alpha\nabla_j\nabla_i u^j + \beta\nabla^2 u_i\\ &\quad + \gamma\nabla_i\nabla_j u^j - (\lambda g_{ij} + T_{ij})u^j = 0.\end{aligned} \quad (2)$$

The only way to have (2) satisfied for any u^i without all vanishing scalars α, β, and γ, is to set $\beta = 0$, whilst $\gamma = -\alpha = \text{const.}$, so that $\alpha\nabla_j\nabla_i u^j + \gamma\nabla_i\nabla_j u^j$ gives the linear combination $\alpha R_{ij}u^j$. R_{ij} being the Ricci tensor.

With these constraints (2) becomes

$$(\alpha R_{ij} - \lambda g_{ij} - T_{ij})u^j = 0, \quad (3)$$

which, in turn, may be satisfied for any u^j if and only if

$$\alpha R_{ij} = T_{ij} + \lambda g_{ij}. \quad (4)$$

Taking the divergence of the two sides gives

$$\frac{1}{2}\alpha\nabla_i R = \nabla_i\lambda. \quad (5)$$

Solving this equation for λ we get $\lambda = (1/2)\alpha R - \Lambda$, for some constant of integration Λ. Substituting λ back into (4) yields

$$\alpha\left(R_{ij} - \frac{1}{2}g_{ij}R\right) + \Lambda g_{ij} = T_{ij}. \quad (6)$$

By choosing $\alpha = 1/8\pi G$ in order to recover the Newtonian limit, we recognize Einstein equations—our constant of integration Λ playing the role of a cosmological constant.

Taking into account the different constraints on the four scalars α, β, γ, and λ that appeared in the general form we chose for the entropy functional in the preceding section, this latter reads

$$\mathcal{S} = \int \mathrm{d}^4x \sqrt{-g}\left[\alpha(\nabla_i u_j \nabla^j u^i - (\nabla_i u^i)^2) + \left(T_{ij} - \frac{1}{2}g_{ij}T + \Lambda g_{ij}\right)u^i u^j\right], \tag{7}$$

where equation $\alpha R = 4\Lambda - T$, that comes from the contraction of Einstein equations (6), is used to eliminate R from the quadratic term in u^i. Using Einstein equations (6) again and integrating by parts [14], one finds (setting $\alpha = 1/8\pi G$)

$$\begin{aligned}\mathcal{S} &= \frac{1}{8\pi G}\int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\nabla_i\left(u^j\nabla_j u^i - u^i\nabla_j u^j\right)\\ &= \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|}n_i\left(u^j\nabla_j u^i - u^i\nabla_j u^j\right),\end{aligned} \tag{8}$$

where h is the determinant of the three-dimensional metric corresponding to the hypersurface $\partial\mathcal{M}$ bounding the integration region $\mathcal{M}$ of space-time, and n_i a unit vector normal to that hypersurface. Using this last expression for the entropy functional, it was shown in [14] that under certain assumptions about the field u^i, the entropy always comes proportional to the horizon of black holes. In Sect. 4 we shall use this form of the functional to investigate its consequences when applied to space-time with the additional constraint that the deformation vector field u^i satisfies a generalized version of the equations of equilibrium of three-dimensional elasticity theory. That generalization is given in the next section.

3 Generalized Elasticity Equations for an Elastic Space-Time

Since we are literally taking space-time to be an elastic medium it is natural to investigate its deformation using a generalization of the elasticity equations.

Recall that in three-dimensional elasticity theory [17] the equations of equilibrium are

$$\partial_j \sigma^{ij} + f^i = 0, \tag{9}$$

where σ^{ij} is the stress tensor subject to the constraint

$$\sigma^{ij} = \sigma^{ji}, \tag{10}$$

and f^i is the total density of external forces and nonelastic forces inside the medium. These equations express the fundamental principle of dynamics applied on the infinitesimal volume element at equilibrium, on which f^i acts. The constraint on σ^{ij} expresses the absence of torques at equilibrium. These equations are written in what is called the stress formulation. One goes to what is called the strain formulation by using Hooke's law for elastic mediums relating the stress tensor to the strain tensor defined by $\varepsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$. For isotropic mediums Hooke's law has the following form $\sigma_{ij} = \mu\delta_{ij}\varepsilon^k_k + \nu\varepsilon_{ij}$, where summation on repeated indices is understood and, μ and ν called the Lamé coefficients, are positive constants in the case of a homogeneous medium characterizing the elastic properties of this latter. In terms of the field u^i, the three-dimensional Hooke's law then writes

$$\sigma_{ij} = \mu\delta_{ij}\partial_k u^k + \nu(\partial_i u_j + \partial_j u_i)/2. \tag{11}$$

Taking space-time to be an isotropic and homogeneous elastic medium we write a four-dimensional generalization of the above three-dimensional Hooke's law in the form $\sigma^{ij} = \mu g^{ij}\nabla_k u^k + \nu\nabla^i u^j + \rho\nabla^j u^i$, where now $i, j = 0, \dots, 3$ and μ, ν and ρ are three positive constants that would characterize the four-dimensional isotropic and homogeneous elastic space-time. On generalizing the three-dimensional constraint (10), however, we learn that we must have $\rho = \nu$ and the four-dimensional Hooke's law takes the following form

$$\sigma^{ij} = \mu g^{ij}\nabla_k u^k + \nu(\nabla^i u^j + \nabla^j u^i). \tag{12}$$

Next, the generalization to space-time of the equation of equilibrium (9) takes, in the absence of nonelastic and exterior forces, the following general covariant form $\nabla_j\sigma^{ij} = 0$. Using (12), this writes in terms of the vector field u^i as

$$\mu\nabla^i\nabla_k u^k + \nu(\nabla_k\nabla^i u^k + \nabla_k\nabla^k u^i) = 0. \tag{13}$$

Finally, since the elastic medium we take space-time to be is not embedded in a higher dimensional manyfold as it is done in the approach of [13], we must impose an additional condition expressing the absence of rigid rotations. In three-dimensional elasticity theory this is done by setting to zero the tensor of local rotations [18] $\omega_{ij} = (\partial_i u_j - \partial_j u_i)/2$. We transcribe this condition to space-time with the following general covariant equation

$$\nabla^i u^j - \nabla^j u^i = 0. \tag{14}$$

The Combination of (13) and (14) gives

$$\nabla_i\nabla_j u^j = -\frac{2\nu}{\mu + 2\nu}R_{ij}u^j = -\frac{16\pi G\nu}{\mu + 2\nu}\left(T_{ij} - \frac{1}{2}T g_{ij} + \Lambda g_{ij}\right)u^i u^j. \tag{15}$$

The second step comes from the use of Einstein equations (6). Note that the above equations may be derived as equations of motion for the field u^i using the following action

$$S = \int \mathrm{d}^4x\, \mathcal{L} = \int \mathrm{d}^4x\, \sqrt{-g}\left[(\nabla_i u^i)^2 - \frac{16\pi G\nu}{\mu + 2\nu}\left(T_{ij} - \frac{1}{2}T g_{ij} + \Lambda g_{ij}\right)u^i u^j\right]. \tag{16}$$

The second term inside the square brackets of the Lagrangian gives the coupling of the field u^i with the energy-momentum tensor of the sources and the cosmological constant. Such a term would play a central role when investigating the deformation field of non-vacuum solutions to Einstein equations such as the Reissner-Nordstøm black holes.

In the next section we shall apply the above generalized equations of equilibrium to space-time in the presence of a black hole to check if our generalization of elasticity yields a deformation of space-time, i.e. a field u^i, that may reproduce familiar results from the thermodynamics of black holes.

4 The Bekenstein-Hawking Entropy Formula

In what follows, we shall apply the final expression (8) of the entropy functional to evaluate the entropy of a Schwarzschild black hole. For that purpose, we need both the metric and the field u^i. Recall that the Schwarzschild metric around a black hole of masse M reads [19]

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \tag{17}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. Thus it remains only to find the deformation four-vector field u^i.

In three-dimensional elasticity theory the deformation vector field u^i is created either from exterior constraints or from defects inside the medium. One then obtains the explicit form of u^i by solving the equations of equilibrium and taking into account the boundary conditions specific to the problem [17].

Taking matter to be space-time defects is the soul idea behind our present study of space-time using four-dimensional elasticity theory. In what follows we shall therefore take the Schwarzschild black hole to be an infinite line defect, passing through the origin (at $r = 0$ in spherical coordinates) and parallel to the time direction, we then compute u^i using the generalized equations of elasticity of Sect. 3.

Now, from the spherical symmetry of the Schwarzschild black hole identified with a line defect parallel to the time direction, it is natural to choose the following Ansatz to represent the deformation vector field around that defect $u^i = (u^0(r), u^1(r), 0, 0)$. To get the exact form of the r-dependence we require u^i to satisfy outside the defect region ($r = 0$) both (13) and (14) for $i, j = 0, \ldots, 3$.

Using the non-vanishing components of the metric connection for Schwarzschild [19],

$$\begin{aligned}
&\Gamma^1_{00} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right), \qquad \Gamma^1_{11} = -\frac{GM}{r(r-2GM)}, \qquad \Gamma^0_{01} = \frac{GM}{r(r-2GM)},\\
&\Gamma^2_{12} = \frac{1}{r}, \qquad \Gamma^1_{22} = -r\left(1 - \frac{2GM}{r}\right), \qquad \Gamma^3_{13} = \frac{1}{r},\\
&\Gamma^1_{33} = -r\left(1 - \frac{2GM}{r}\right)\sin^2\theta, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta},
\end{aligned} \tag{18}$$

reveals after a straightforward calculation of the various covariant derivatives that for $i, j = 2, 3$ (13) and (14) are trivially satisfied. For $i, j = 0, 1$, however, they give rise to the following system of differential equations

$$\begin{aligned}
\mu\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}^2u^0}{\mathrm{d}r^2} + \frac{2}{r-2GM}\frac{\mathrm{d}u^0}{\mathrm{d}r}\right) &= 0,\\
(\mu + 2\nu)\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}^2u^1}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}u^1}{\mathrm{d}r} - \frac{2}{r^2}u^1\right) &= 0,\\
\left(1 - \frac{2GM}{r}\right)\left(\frac{\mathrm{d}u^0}{\mathrm{d}r} + \frac{2GMu^0}{r^2 - 2GMr}\right) &= 0,
\end{aligned} \tag{19}$$

to which the general solution for positive μ and ν and $r \neq 2GM$ is

$$u^0 = \frac{Br}{r-2GM}, \qquad u^1 = \frac{C}{r^2} + Dr. \tag{20}$$

B, C, and D are three constants of integration.

Demanding a deformation vector field that is bounded at infinity, we set D equal to zero. On the other hand, since this field is time independent we may absorb the constant B and set it equal to one in the time component of the field u^i by making a global rescaling of the field together with a redefinition of the time coordinate $\mathrm{d}t \to B^{-1}\,\mathrm{d}t$ in the infinite time integral of (8). Hence, we rewrite the final solution for the field u^i to be used in integral (8) as follows

$$u^0 = \frac{r}{r-2GM}, \qquad u^1 = \frac{C'}{r^2}, \qquad u^2 = 0, \qquad u^3 = 0. \tag{21}$$

We see that the field u^i is defined everywhere except at $r = 0$ and $r = 2GM$ where it presents two singularities. The event horizon is a time-like singularity while at the origin we have a space-like singularity.

Now back to integral (8). Although the integral decomposes into four parts

$$\begin{aligned}\mathcal{S} = &-\frac{1}{8\pi G}\int \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\, r^2\sin\theta\Big(u^i\nabla_i u^0 - u^0\nabla_i u^i\Big)\\ &+\frac{1}{8\pi G}\int \mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\phi\, r^2\sin\theta\Big(u^i\nabla_i u^1 - u^1\nabla_i u^i\Big)\\ &+\frac{1}{8\pi G}\int \mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\phi\, r^2\sin\theta\Big(u^i\nabla_i u^2 - u^2\nabla_i u^i\Big)\\ &+\frac{1}{8\pi G}\int \mathrm{d}t\,\mathrm{d}r\,\mathrm{d}\theta\, r^2\sin\theta\Big(u^i\nabla_i u^3 - u^3\nabla_i u^i\Big),\end{aligned} \tag{22}$$

where use of the metric (17) has been made, only the second integral survives when we plug-in the solution (21) for u^i. Indeed, the integrands in the last two integrals vanish trivially because of the general form $(u^0(r), u^1(r), 0, 0)$ of the field u^i while the integrand in the first integral vanishes only because of the precise form of the solution (21). We are then left with the second integral which is an integral over the hypersurface normal to the radial direction and bounding the whole region $0 \leq r < \infty$ of space-time. To perform the infinite time integral we make an analytic continuation and use the Euclideanized time $\tau = it$ which is related to the temperature T of the corresponding thermal system by $\tau = \beta = 1/T$. Using (21) and the metric connection (18) to calculate the covariant derivatives, and then taking the limit $r \to \infty$ (on the hypersurface), we find

$$\begin{aligned}\mathcal{S} &= \lim_{r\to\infty}\frac{1}{8\pi G}\int_0^{1/T}\mathrm{d}\tau\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\int_0^{2\pi}\mathrm{d}\phi\left(\frac{GMr}{r-2GM} - \frac{C'(2r-3GM)}{r^3(r-2GM)}\right)\\ &= \frac{M}{2T}.\end{aligned} \tag{23}$$

Note that it is the component of the field u^i that is singular on the horizon that contributes to entropy. Substituting this result into the standard thermodynamical relation $\mathrm{d}E = T\,\mathrm{d}\mathcal{S}$, where E is the energy of the black hole related to its mass by $E = M$ (in natural units), we obtain that $\mathrm{d}M = T\,\mathrm{d}(M/2T)$ and hence that the black hole's temperature is a function of its mass M, that is $T = T(M)$. Taking this into account in performing the differential, we arrive at the differential equation $\mathrm{d}T/\mathrm{d}M + T/M = 0$, which solves for $T(M)$ up to an integration constant to give $T = \mathrm{const}/M$. By dimensional analysis we learn that in our system of units where the Boltzmann constant is set equal to unity $k_B = 1$, the constant of integration must be proportional to the inverse of Newton's constant G, that is

$$T = \frac{\gamma}{GM}, \tag{24}$$

for some proportionality numerical factor γ. Identity (24) agrees (with its inverse mass-dependence) exactly with the Hawking temperature for a Schwarzschild black hole for the value $\gamma = 1/8\pi$.

Finally, substituting (24) back into (23) we get the final value of the entropy $\mathcal{S} = GM^2/2\gamma$. In terms of the Schwarzschild radius $R = 2GM$, this reads

$$\mathcal{S} = \frac{R^2}{8\gamma G} = \frac{A}{32\pi\gamma G}, \tag{25}$$

where $A = 4\pi R^2$ is the area of the event horizon. For the value $\gamma = 1/8\pi$, our entropy is in exact agreement with the Bekenstein-Hawking entropy formula $\mathcal{S} = A/4G$ for Schwarzschild black holes.

5 The 'Elastic' Mass of the Schwarzschild Black Hole

Now that we have found the deformation vector field u^i around the Schwarzschild black hole we may ask if, being viewed as an infinite line defect in space-time, it is possible to recover its mass using the information we gained about the elastic deformation it caused to space-time through the field u^i. In three-dimensional elasticity the component σ^{ij} of the stress-tensor represents a force (per unit surface) parallel to the direction i and acting on the surface normal to the direction j. Since in four dimensions the hypersurface normal to the time direction is the spatial volume it is natural in our generalized elasticity to interpret, in analogy to the Maxwell stress-tensor T^{00} of electromagnetism, the component σ^{00} of the generalized stress-tensor as the energy density associated with the elastic deformation of space-time. As such, we expect to recover the energy associated with the black hole, i.e. the black hole's mass, by integrating σ^{00} over the whole time-like surface Σ_0 normal to the time direction, that is over all space around the black hole.

Plugging the two non-vanishing components (20) (with $D = 0$) of u^i into the expression (12) of σ^{ij} we obtain $\sigma^{00} = 2\nu\nabla^0 u^0$. Using the metric (17) and the connection (18) to find explicitly $\nabla^0 u^0\,\mathrm{d}\Sigma_0$ gives after integration

$$E = 2\nu \int \nabla^0 u^0\,\mathrm{d}\Sigma_0 = -2\nu \int_0^\infty \frac{CGM\mathrm{d}r}{r^2 - 2GMr} \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi = \nu C. \tag{26}$$

Identifying this finite result with mass M we learn that it is indeed possible to recover, using elasticity, the mass of the black hole provided that $\nu C = M$. Now since the constant ν is the analog for space-time of one of the Lamé coefficients of three dimensional elastic bodies, it must only characterize the properties of space-time and must not depend on the mass of the black hole which is just a defect in this medium. Hence, we deduce that the constant of integration C must be proportional to M, a result which is amply satisfactory for the two following reasons. On one hand, it is in accord with one's intuition to have this proportionality since we expect a zero deformation field in the absence of the defect that gave rise to it, i.e. for $M = 0$. On the other hand, we also expect only a linear dependence on the mass M because of the linearity of the equations of equilibrium (13) and (14).

6 Discussion and Conclusion

We saw that assuming space-time to be an elastic continuum permits not only to arrive at Einstein equations but also to recover a cosmological constant as an integration constant in accord with what is already found in [14] using different arguments. For an interpretation of the appearance of that constant in the functional (7) and a discussion on its order of magnitude see [14, 20, 21].

In Sect. 3, we gave constraints on the deformation of space-time in vacuum that are nothing but generalized equations from three-dimensional elasticity theory. Using Einstein equations we saw that it is possible to construct a general Lagrangian to describe the elastic deformations of space-time even for non-vacuum solutions, that is in the presence of non-vanishing external fields.

In Sects. 4 and 5 we made, as a test for our generalized elasticity and the entropy functional (7), an application to black holes. We saw that for the simple case of a Schwarzschild black hole, viewed as an infinite line defect in space-time, the deformation of space-time surrounding the defect, reproduces the energy content of the black hole, i.e. its mass, and generates an entropy that agrees in its form with the one conjectured by Bekenstein on the basis of the classical properties of black holes. Furthermore, we saw that for a precise value of the numerical factor γ arising as a constant of integration this later result agrees exactly with the value found by Hawking on the basis of quantum field theory in curved space-time. It is then remarkable to arrive at such a result without making any arbitrary assumption on the field u^i. The only guidance we had in finding the deformation vector field u^i was the generalization of the classical equations of three-dimensional elasticity. How then, one may wonder, is it possible to reproduce a result that agrees with a quantity that is essentially quantum mechanical using a calculation that is based on the classical concepts of elasticity theory. Speculating upon a way through this seemingly paradoxical situation we provide an answer that is twofold. First, the use of the analytic continuation $t \to it = \tau$ to perform the time integral in (23) made it possible to use the β-periodic Euclidean time which is inherent to quantum thermal systems. The second hint lies in (25) that gives the entropy of the black hole. Indeed, this formula gives the value of $\mathcal{S}$ up to a numerical constant factor γ, a result reminiscent of the one appearing in a loop quantum gravity (LQG) derivation of that entropy [22]. In LQG, the entropy of a black hole is given up to a certain constant numerical factor called the Immirzi parameter, whose origin comes from the evaluation of surfaces in quantum geometry. Thus, although our starting point was to adapt the classical concept of deformation vector fields of three-dimensional elasticity to four dimensions, its nature and origin in space-time is rather deeply quantum mechanical. But then the following question arises: what is the quantum origin of this field? We hope that further investigation could lead to an understanding of the mechanism behind this 'quantum' elasticity of space-time and may even be very promising in giving a new way to investigate the quantum description of gravity.

Finally, Although the calculations involved may become tedious, our method may easily be applied without any conceptual difficulty to get the entropy of different and more complicated types of black holes, such as the charged Reissner-Nordstrøm black hole or the rotating Kerr black hole. Thus, identifying space-time with an elastic medium allows one not only to recover well-known results more quickly and easily, but also to look for new possibilities to tackle some of the modern outstanding problems in the study of space-time such as the quantum origin of gravity and that of dark energy in cosmology.

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